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Existence of Periodic Solutions of $(2n+1)^{\text{th}}$-Order Ordinary Differential Equations

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Abstract—This note concerns a periodic solution of a class of odd order nonlinear ordinary differential equations. An existence theorem of periodic solutions is obtained.

Keywords—Odd order differential equation, Existence, Periodic solution.

The purpose of this paper is to investigate the problem on the existence of a periodic solution for nonlinear ordinary differential equations

$$x^{(2n+1)} + f\left(t, x, x', \dots, x^{(2n)}\right) = 0. \tag{1}$$

It will always be assumed that the function $f : R \times R^{2n+1} \to R$ is continuous and 2π-periodic with respect to t.

One of the important topics in the qualitative theory of ordinary differential equations is the study of the existence and/or uniqueness of a periodic solution. In recent years, there has been increasing interest in the existence problem for the high-order differential equations, for example, [1–9]. In [7], the equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + g(t, x) = 0 \tag{2}$$

was discussed. Applying the continuation theorem of Mawhin, the existence of periodic solutions for (2) was obtained.

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This note continues the work of [7]. Let a_i, $i = 0, 1, \dots, 2n$, be positive given constants. Let D_i be a subset of R^{i+1} as follows:

$$D_i = \left\{(x_0, x_1, \dots, x_i) \in R^{i+1} : |x_j| < a_j,\ j = 0, 1, \dots, i\right\}, \qquad i = 0, 1, \dots, 2n.$$

Define

$$f_{i+1}(t, x_{i+1}, \dots, x_{2n}) = \sup_{(x_0, \dots, x_i) \in D_i} |f(t, x_0, \dots, x_{2n})|.$$

From now on, we assume that the following hold:

(H1) there exist positive constants b_0, c_i, $i = 0, 1, \dots, 2n$, to satisfy, as $|x_i| \geq a_i$ and $(x_{i+1}, \dots, x_{2n}) \in R^{2n-i}$,

$$\left|\frac{f_i}{x_i}\right| \leq c_i, \qquad i = 1, 2, \dots, 2n, \tag{3}$$

for all x_0 with $|x_0| \geq a_0$ and for all $(x_1, \dots, x_{2n}) \in R^{2n}$,

$$b_0 \leq \frac{f}{x_0} \leq c_0 \qquad \text{or} \qquad -c_0 \leq \frac{f}{x_0} \leq -b_0; \tag{4}$$

(H2) the constants b_0 and c_i suit the inequality

$$\left(1 + \frac{c_0}{b_0}\right) \sum_{i=1}^{2n} 2^i c_i < 1.$$

THEOREM A. *If Conditions (H1) and (H2) above are satisfied, then equation (1) has at least one 2π-periodic solution.*

REMARK. We do not impose the smoothness on the function f. This is different from [6].

To prove the theorem, the Wirtinger inequality is needed.

LEMMA 1. *(See [10].) Let x be a continuously differentiable function, and 2π-periodic in t. Assume that*

$$\int_0^{2\pi} x(t)\, \mathrm{d}t = 0.$$

Then,

$$\int_0^{2\pi} x^2(t)\, \mathrm{d}t \leq \int_0^{2\pi} \left(x'(t)\right)^2\, \mathrm{d}t.$$

We first divided f into the sum of certain terms. Define

$$g_0(t, x_0, \dots, x_{2n}) = \begin{cases} x_0^{-1} f(t, x_0, x_1, \dots, x_{2n}), & |x_0| \geq a_0, \\ a_0^{-1} f(t, a_0, x_1, \dots, x_{2n}), & 0 < x_0 < a_0, \\ -a_0^{-1} f(t, -a_0, x_1, \dots, x_{2n}), & -a_0 < x_0 < 0, \\ \delta b_0, & x_0 = 0, \end{cases} \tag{5}$$

where $\delta = \operatorname{sgn}(f/x_0)$. By (5),

$$b_0 \leq g_0 \leq c_0 \quad \text{or} \quad -c_0 \leq g_0 \leq -b_0, \qquad \text{for all } (t, x_0, \dots, x_{2n}) \in R \times R^{2n+1}.$$

Let

$$h_0(t, x_0, \dots, x_{2n}) = f(t, x_0, \dots, x_{2n}) - x_0 g_0(t, x_0, \dots, x_{2n}). \tag{6}$$

It is clear that as $(t, x_0, \dots, x_{2n}) \in O_0 = \{(t, x_0, \dots, x_{2n}) \in R \times R^{2n+1} : |x_0| \geq a_0\}$,

$$h_0(t, x_0, \dots, x_{2n}) = 0, \tag{7}$$

and for all $(t, x_0, \dots, x_{2n}) \in R \times R^{2n+1}$,

$$|h_0(t, x_0, \dots, x_{2n})| \le 2 f_1(t, x_1, \dots, x_{2n}). \tag{8}$$

Set

$$O_i = \bigcup_{j=0}^{i} \left\{ (t, x_0, \dots, x_{2n}) \in R \times R^{2n+1} : |x_j| \ge a_j \right\}.$$

Inductively, if h_i is determined and it satisfies that as $(t, x_0, \dots, x_{2n}) \in O_i$

$$h_i(t, x_0, \dots, x_{2n}) = 0, \tag{9}$$

and for all $(t, x_0, \dots, x_{2n}) \in R \times R^{2n+1}$,

$$|h_i(t, x_0, \dots, x_{2n})| \le 2^{i+1} f_{i+1}(t, x_{i+1}, \dots, x_{2n}), \tag{10}$$

then we will determine h_{i+1} up to h_{2n}. Define

$$g_{i+1}(t, x_0, \dots, x_{2n}) = \begin{cases} x_{i+1}^{-1} h_i(t, x_0, x_1, \dots, x_{2n}), & |x_{i+1}| \ge a_{i+1}, \\ a_{i+1}^{-1} h_i(t, x_0, \dots, x_i, a_{i+1}, x_{i+2}, \dots, x_{2n}), & 0 < x_{x+1} < a_{x+1}, \\ -a_{i+1}^{-1} f(t, x_0, \dots, x_i, -a_{i+1}, x_{i+2}, \dots, x_{2n}), & -a_{i+1} < x_{i+1} < 0, \\ 0, & x_{i+1} = 0. \end{cases} \tag{11}$$

Using (3) and (10), we obtain

$$|g_{i+1}| \le 2^{i+1} c_{i+1}$$

on $R \times R^{2n+1}$. Let

$$h_{i+1}(t, x_0, \dots, x_{2n}) = h_i(t, x_0, \dots, x_{2n}) - x_{i+1} g_{i+1}(t, x_0, \dots, x_{2n}). \tag{12}$$

The definitions of f_{i+2} and g_{i+1}, respectively, and (9) and (10) imply that as $(t, x_0, \dots, x_{2n}) \in O_{i+1}$

$$h_{i+1}(t, x_0, \dots, x_{2n}) = 0, \tag{13}$$

and on $R \times R^{2n+1}$,

$$|h_{i+1}(t, x_0, \dots, x_{2n})| \le 2^{i+2} f_{i+2}(t, x_{i+2}, \dots, x_{2n}). \tag{14}$$

Sum the above key points as the following lemma.

LEMMA 2. *The function f may be written into*

$$f(t, x_0, \dots, x_{2n}) = \sum_{j=0}^{2n} x_j g_j(t, x_0, \dots, x_{2n}) + h_{2n}(t, x_0, \dots, x_{2n}). \tag{15}$$

Moreover, g_j and h_{2n} satisfy

$$b_0 \le g_0 \le c_0 \qquad \text{or} \qquad -c_0 \le g_0 \le -b_0, \tag{16}$$

$$|g_j| \le 2^j c_j, \qquad j = 1, 2, \dots, 2n, \tag{17}$$

$$|h_{2n}| \le c_{2n+1}, \tag{18}$$

where

$$c_{2n+1} = 2^{2n+1} \sup_{t \in R, |x_{2n}| < a_{2n}} f_{2n}(t, x_{2n}).$$

Now we begin to prove Theorem A. From Lemma 2 finding a solution of (1) is equivalent to one of the equation

$$x^{(2n+1)} + \sum_{j=0}^{2n} x^{(j)} g_j\left(t, x, x', \dots, x^{(2n)}\right) + h_{2n}\left(t, x, x', \dots, x^{(2n)}\right) = 0. \tag{19}$$

From (5) and (11) g_j cannot be continuous at 0, but $x^{(j)} g_j$ are continuous. So h_j are continuous by (6) and (12). We introduce auxiliary equations as follows:

$$x^{(2n+1)} + b_0 x + \lambda\left(\sum_{j=0}^{2n} x^{(j)} g_j\left(t, x, x', \dots, x^{(2n)}\right) - b_0 x + h_{2n}\left(t, x, x', \dots, x^{(2n)}\right)\right) = 0, \quad \lambda \in [0,1]. \tag{20}$$

From (16), without loss of generality, assume that $g_0 > 0$. If not, we replace b_0 by $-b_0$ in (20).

LEMMA 3. *There exists a constant $M > 0$ such that each possible 2π-periodic solution $x_\lambda(t)$ of (20) satisfies*

$$\|x_\lambda(t)\| \le M,$$

for all $\lambda \in [0,1]$, where $\|\cdot\|$ is usual C^{2n} norm.

PROOF. Simply, we write the solution x_λ of (20) as x. Multiplying two sides of (20) by x and integrating from 0 to 2π in t, we have

$$\int_0^{2\pi} x^2 \,\mathrm{d}t \le \frac{1}{b_0} \sum_{i=1}^{2n} 2^i c_i \int_0^{2\pi} |x| \cdot \left|x^{(i)}\right| \,\mathrm{d}t + c_{2n+1} \int_0^{2\pi} |x| \,\mathrm{d}t. \tag{21}$$

Here, Lemma 2 and $\int_0^{2\pi} x \cdot x^{(2n+1)} \,\mathrm{d}t = 0$ are used.

It is easy to prove that $x^{(i)}$, $i = 1, 2, \dots, 2n$, satisfy the conditions of Lemma 1. From Lemma 1 and the Schwarz inequality

$$\left(\int_0^{2\pi} x^2 \,\mathrm{d}t\right)^{1/2} \le \frac{1}{b_0}\left(\int_0^{2\pi}\left(x^{(2n)}\right)^2 \mathrm{d}t\right)^{1/2} \sum_{i=1}^{2n} 2^i c_i + M_1, \tag{22}$$

where M_1 is positive constant depending only on c_i. Below M_i, $i = 2, 3, \dots$, similarly, are understood.

On the basis of (20), Lemma 2, Lemma 1, and Schwarz inequality,

$$\begin{aligned}
\int_0^{2\pi}\left(x^{(2n)}\right)^2 \mathrm{d}t &= -\int_0^{2\pi} x^{(2n-1)} x^{(2n+1)} \,\mathrm{d}t \\
&\le \lambda \sum_{i=0}^{2n} 2^i c_i \int_0^{2\pi}\left|x^{(2n-1)} x^{(i)}\right| \mathrm{d}t + c_{2n+1} \int_0^{2\pi}\left|x^{(2n-1)}\right| \mathrm{d}t \\
&\qquad \left(\text{by } \int_0^{2\pi} x \cdot x^{(2n-1)} \,\mathrm{d}t = 0\right) \\
&\le \sum_{i=1}^{2n} 2^i c_i \int_0^{2\pi}\left(x^{(2n)}\right)^2 \mathrm{d}t + c_0\left(\int_0^{2\pi} x^2 \,\mathrm{d}t\right)^{1/2}\left(\int_0^{2\pi}\left(x^{(2n)}\right)^2 \mathrm{d}t\right)^{1/2} \\
&\quad + M_2\left(\int_0^{2\pi}\left(x^{(2n)}\right)^2 \mathrm{d}t\right)^{1/2},
\end{aligned}$$

which with (22) and Assumption (H2) leads to the inequality

$$\left(\int_0^{2\pi}\left(x^{(2n)}\right)^2 \mathrm{d}t\right)^{1/2} \le M_3. \tag{23}$$

Hence, from (22), (23), and Lemma 1,

$$\int_0^{2\pi} \left(x^{(i)}\right)^2 \mathrm{d}t \le M_4^2, \qquad i = 0, 1, \ldots, 2n, \tag{24}$$

which demonstrate that there exist $t_i \in [0, 2\pi]$, $i = 0, 1, \ldots, 2n$, to suit

$$x^{(i)}(t_i) \le \frac{M_4}{\sqrt{2\pi}}, \qquad i = 0, 1, \ldots, 2n. \tag{25}$$

For any $t \in [0, 2\pi]$, according to (23) and (24),

$$\begin{aligned} \left|x^{(i)}(t)\right| &\le \left|x^{(i)}(t_i)\right| + \left|\int_{t_i}^{t} x^{(i+1)}\,\mathrm{d}t\right| \\ &\le \frac{M_4}{\sqrt{2\pi}} + \sqrt{2\pi}\left(\int_0^{2\pi} \left(x^{(i+1)}\right)^2 \mathrm{d}t\right)^{1/2} \le M_5, \qquad i = 0, 1, \ldots, 2n-1. \end{aligned} \tag{26}$$

In addition,

$$\begin{aligned} \left|x^{(2n)}(t)\right| &\le \left|x^{(2n)}(t_{2n})\right| + \lambda \sum_{j=0}^{2n} 2^j c_j \int_0^{2\pi} \left|x^{(j)}\right| \mathrm{d}t \\ &\quad + \lambda 2^{2n+2} c_{2n+1} + (1-\lambda) b_0 \int_0^{2\pi} |x|\,\mathrm{d}t \\ &\le M_6 + 2^{2n} c_{2n} \int_0^{2\pi} \left|x^{(2n)}\right| \mathrm{d}t \qquad \text{(by (26))} \\ &\le M_6 + 2^{2n}\sqrt{2\pi}\, c_{2n} \left(\int_0^{2\pi} \left(x^{(2n)}\right)^2 \mathrm{d}t\right)^{1/2} \le M_7 \qquad \text{(by (24))}. \end{aligned} \tag{27}$$

According to (26) and (27), we derive the estimate in Lemma 3. ∎

LEMMA 4. *For any $\alpha \neq 0$, the equation*

$$x^{(2n+1)} + \alpha x = 0$$

has only trivial 2π-periodic solution.

The proof of Lemma 4 is easy.

Let $\Omega = \{u \in C(R, R^{2n+1}) : u(t + 2\pi) = u(t),\ \|u\| \le M + 1\}$, where $\|u\| = \sum_{j=1}^{2n+1} \sup_{0\le t\le 2\pi} |u_j(t)|$. Rewrite (20) as follows:

$$u' = F_\lambda(t, u) = F_\lambda(t + 2\pi, u), \tag{28}$$

where $u = (x, x', \ldots, x^{(2n)})^\top$, $F_\lambda(t, u) = (x', x'', \ldots, x^{(2n)}, (1-\lambda)b_0 x + \lambda f(t, x, x', \ldots, x^{(2n)}))^\top$. By Lemma 3, for each $\lambda \in [0, 1]$, every possible 2π-periodic solution u of (28) satisfies $u \notin \partial\Omega$. By Lemma 4, as $\lambda = 0$, (28) has only trivial solution. Thus, from [5] (Theorem 1 P_{33}), as $\lambda = 1$, (28) has at least one 2π-periodic solution, that is, (1) has a 2π-periodic solution. Theorem A is proved. ∎

From Theorem A we also have the following theorem.

THEOREM B. *Assume that f satisfies*

$$|f(t, x_0, x_1, \ldots, x_{2n}) - f(t, y_0, y_1, \ldots, y_{2n})| \le \sum_{i=0}^{2n} c_i |x_i - y_i|,$$

for any $(t, x_0, x_1, \ldots, x_{2n})$ and $(t, y_0, y_1, \ldots, y_{2n}) \in R \times R^{2n+1}$, where $c_0 > 0$ and $c_i \ge 0$, $i = 1, 2, \ldots, 2n$, are constants and there exists a constant $b_0 > 0$ such that $b_0 \le |f_{x_0}| \le c_0$ on $R \times R^{2n+1}$ and the following inequality holds:

$$\left(1 + \frac{c_0}{b_0}\right) \sum_{i=1}^{2n} 2^i c_i < 1.$$

Then, equation (1) has at least one 2π-periodic solution.

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